

TORSION PAIRS IN CLUSTER TUBES

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ABSTRACT. We give a complete classification of torsion pairs in the cluster categories associated to tubes of finite rank. The classification is in terms of combinatorial objects called Ptolemy diagrams which already appeared in our earlier work on torsion pairs in cluster categories of Dynkin type A. As a consequence of our classification we establish closed formulae enumerating the torsion pairs in cluster tubes, and obtain that the torsion pairs in cluster tubes exhibit a cyclic sieving phenomenon.

Dedicated to Idun Reiten on the occasion of her 70th birthday.

1. INTRODUCTION

Torsion pairs form a fundamental and important aspect of representation theory. Their definition for abelian categories goes back to a paper by S. Dickson [9] from the mid 1960s. Torsion pairs play a particularly prominent role in the context of tilting theory. The latter has been introduced in the early 1980s in the seminal papers by S. Brenner and M. Butler [6] and by D. Happel and C. Ringel [10] and has remained a key topic in the representation theory of finite-dimensional algebras ever since.

Many important developments in modern representation theory take place not in the module categories themselves but in related triangulated categories like stable and derived categories or cluster categories. The latter have been introduced in [7] and provide a highly successful categorification of Fomin and Zelevinsky's cluster algebras.

In this paper we consider cluster categories coming from tubes and as main result give a complete classification of the torsion pairs in these *cluster tubes*. This classification is in terms of certain combinatorial objects called Ptolemy diagrams which have been introduced by P. Ng [17] and already occurred in the classifications of torsion pairs in cluster categories of Dynkin type A [12] and A_∞ [17].

It turns out that for a given rank there are only finitely many torsion pairs in the cluster tube and as a second main result we provide a complete enumeration for torsion pairs in cluster tubes, i.e. we establish a formula for the generating function

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and from that deduce a closed formula for the number of torsion pairs in cluster tubes of a given rank.

Tubes arise naturally and frequently in representation theory, not least as components of the Auslander-Reiten quivers of many finite dimensional algebras. Tubes are stable translation quivers of the form $\mathbb{Z}A_\infty/(n)$ for a natural number n , i.e. they have the shape of a cylinder which is infinite on one side. For details on how tubes arise in the Auslander-Reiten theory of algebras of tame representation type we refer to C.M. Ringel's book [19, Chapter 3]. Taking the additive hull of the mesh category of a tube as above one obtains an abelian category T_n , called a *tube category of rank n* . It naturally occurs in the representation theory of extended Dynkin quivers \tilde{A} ; namely, the category T_n is equivalent to the category of nilpotent representations of a cyclic quiver with n vertices and edges cyclically oriented (cf. [19, sec. 3.6 (6)]).

In a recent paper, K. Baur, A. Buan and R. Marsh [2] have classified torsion pairs in this abelian category T_n . On the other hand, our results in this paper concern the triangulated cluster category (to be defined below) attached to the tube category. Although both categories have the same Auslander-Reiten quiver it turns out that their torsion theories are different. In particular, our results in this paper do not overlap with the results in [2].

The abelian category T_n has many nice properties, in particular it is hereditary and Hom-finite. This allows to form, as in [7], the cluster category attached to T_n as the orbit category

$$\mathsf{C}_n := D^b(\mathsf{T}_n)/(\tau^{-1} \circ \Sigma)$$

where $D^b(\mathsf{T}_n)$ is the bounded derived category of the hereditary abelian category T_n and τ and Σ are the Auslander-Reiten translation and the suspension on $D^b(\mathsf{T}_n)$, respectively. From a result of B. Keller [14] it follows that this orbit category is a triangulated category.

This triangulated category C_n is called the *cluster tube of rank n* .

We shall prove the following result in this paper which provides a complete classification of torsion pairs in cluster tubes. In fact, as an important step on the way we show in Proposition 4.1 that for each torsion pair in C_n precisely one of the subcategories X and X^\perp has only finitely many indecomposable objects.

For the definition of Ptolemy diagrams and levels we refer to Section 3 and for wings to Section 4 below.

Theorem 1.1. *There are bijections between the following sets:*

- (i) *Torsion pairs $(\mathsf{X}, \mathsf{X}^\perp)$ in the cluster tube C_n such that X has only finitely many indecomposable objects.*
- (ii) *n -periodic Ptolemy diagrams \mathcal{X} of the ∞ -gon such that all arcs in \mathcal{X} have length at most n .*
- (iii) *Collections $\{[(i_1, j_1)], [W_1)], \dots, [(i_r, j_r)], [W_r)]\}$ of pairs consisting of vertices $[(i_\ell, j_\ell)]$ of level $\leq n-1$ in the AR-quiver of C_n and subsets $[W_\ell] \subseteq W[(i_\ell, j_\ell)]$ of their wings such that for any different $k, \ell \in \{1, \dots, r\}$ we have*

$$\Sigma W[(i_k, j_k)] \cap W[(i_\ell, j_\ell)] = \emptyset$$

and the n -periodic collection W_ℓ of arcs corresponding to $[W_\ell]$ is a Ptolemy diagram (in which every arc is overarched by some arc from the collection corresponding to $[(i_\ell, j_\ell)]$).

Along the way we also obtain an independent proof of the following result of A. Buan, R. Marsh and D. Vatne.

Corollary 1.2 ([8]). *The cluster tubes C_n do not contain any cluster tilting objects.*

One of the outcomes of the above classification result is that for a given rank n there are only finitely many torsion pairs in the cluster tube C_n . In the final section we count the number of torsion pairs and obtain the following enumeration result.

Theorem 1.3. *The number of torsion pairs in the cluster tube C_n is equal to*

$$\tau_n = \sum_{\ell \geq 0} 2^{\ell+1} \binom{n-1+\ell}{\ell} \binom{2n-1}{n-1-2\ell}.$$

In Section 5 we also present refined counts and determine the number of torsion pairs up to Auslander-Reiten translation. Moreover, we show that the set of torsion pairs in the cluster tube C_n together with the cyclic group generated by the Auslander-Reiten translation exhibits a cyclic sieving phenomenon.

The paper is organised as follows. In Section 2 we review some fundamentals on torsion pairs in triangulated categories, building on the paper by O. Iyama and Y. Yoshino [13]. In Section 3 we set the scene by recalling a recent geometric model introduced by K. Baur and R. Marsh [3] and by connecting it to another combinatorial model via arcs in an ∞ -gon, as introduced in [11]. Section 4 then contains the proofs of our main classification results. Finally, in Section 5 we prove the enumeration results for torsion pairs in cluster tubes and explain the cyclic sieving phenomenon.

2. TORSION THEORY IN TRIANGULATED CATEGORIES

Torsion theory in abelian categories has been introduced by S. Dickson [9] in 1966. More recently, this concept has been taken to the world of triangulated categories by O. Iyama and Y. Yoshino [13]. In this section we briefly review some fundamentals.

For the rest of the paper we always assume that triangulated categories C are K -linear over a fixed field K , and that they are Krull-Schmidt and Hom-finite (i.e. all morphism spaces are finite dimensional).

Moreover, any subcategory of a triangulated category is always supposed to be full, and closed under isomorphisms, direct sums and direct summands.

Definition 2.1. A *torsion pair* in a triangulated category C is a pair (X, Y) of subcategories such that

- (i) the morphism space $\text{Hom}_C(x, y)$ is zero for all $x \in X, y \in Y$,
- (ii) each object $c \in C$ appears in a distinguished triangle $x \rightarrow c \rightarrow y \rightarrow \Sigma x$ with $x \in X, y \in Y$.

Examples of such torsion pairs in the triangulated situation are given by the t-structures of Beilinson, Bernstein, and Deligne [1] where, additionally, one assumes $\Sigma X \subseteq X$, and by the co-t-structures of Bondarko [5] and Pauksztello [18] where, additionally, one assumes $\Sigma^{-1}X \subseteq X$. It is not hard to deduce from the definition that a torsion pair (X, Y) is determined by one of its entries, namely one has that

$$Y = X^\perp := \{c \in C \mid \text{Hom}_C(x, c) = 0 \text{ for each } x \in X\},$$

$$\text{and } X = {}^\perp Y := \{c \in C \mid \text{Hom}_C(c, y) = 0 \text{ for each } y \in Y\}.$$

Moreover, as pointed out in [13, rem. after def. 2.2], for any torsion pair (X, Y) , both subcategories are closed under extensions and X is contravariantly finite in C and Y is covariantly finite in C .

Then we have the following useful characterisation of torsion pairs, due to O. Iyama and Y. Yoshino.

Proposition 2.2. ([13, prop. 2.3]) *Let X be a contravariantly finite subcategory of C . Then (X, X^\perp) is a torsion pair if and only if $X = {}^\perp(X^\perp)$.*

Similarly, let Y be a covariantly finite subcategory of C . Then $({}^\perp Y, Y)$ is a torsion pair if and only if $Y = ({}^\perp Y)^\perp$.

3. A GEOMETRIC MODEL FOR CLUSTER TUBES

A geometric model for the cluster category associated to a tube has been proposed by K. Baur and R. Marsh in [3]. Instead of using their annulus model directly we prefer (as the authors also partially do in [3]) to work with an infinite cover of it, namely arcs in an ∞ -gon. The latter has vertices indexed by the integers; an arc in the ∞ -gon is a pair (i, j) of integers with $j - i \geq 2$, and i and j are called endpoints of the arc.

It is useful to think of an arc geometrically as a curve between two integers on the number line as in the following picture.



This arc model has been extensively used in earlier papers, see e.g. [17], [11]; of particular importance is the situation where two arcs cross.

Definition 3.1. Two arcs (i, j) and (k, l) of the ∞ -gon are said to *cross* if either $i < k < j < l$ or $k < i < l < j$.

This definition precisely corresponds to the geometric intuition in that two arcs can be drawn as non-crossing curves if and only if they do not cross in the sense of Definition 3.1. Note that two arcs which only meet in some of their endpoints are not crossing.

The Auslander-Reiten quivers of the tube category T_n of rank n and the corresponding cluster tube C_n both have the shape of a cylinder of circumference n , infinite to one side. More precisely, it is of the form $\mathbb{Z}A_\infty/(\tau^n)$ where τ is the Auslander-Reiten translation. The indecomposable objects are arranged in the quiver as shown

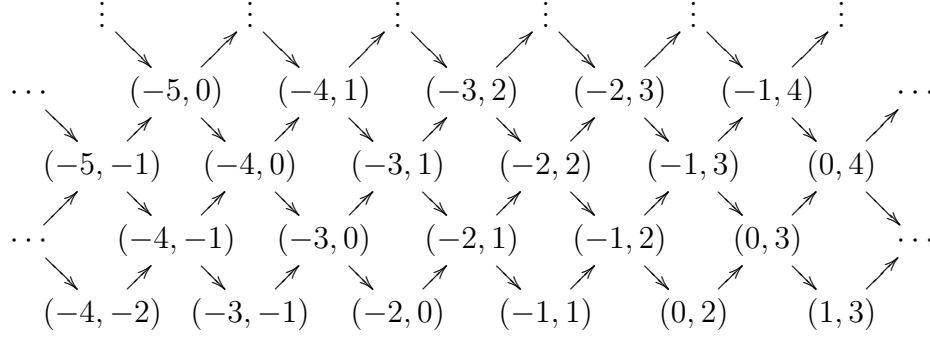


FIGURE 1. Coordinate system on the AR quiver

in Figure 1, where the vertices are identified after n steps. We will frequently use the standard coordinate system given in Figure 1.

The Auslander-Reiten translation τ is on coordinates just given by $\tau : (i, j) \mapsto (i-1, j-1)$, i.e. for the AR-quiver of \mathbf{T}_n and \mathbf{C}_n this implies that the coordinates (i, j) and $(i-n, j-n)$ have to be identified for all i, j , i.e. both coordinates simultaneously have to be taken modulo n .

It is easy to see that there is a bijection between the vertices (i, j) in the above coordinate system on the translation quiver $\mathbb{Z}A_\infty$ and arcs of the ∞ -gon.

There is a covering map from the translation quiver $\mathbb{Z}A_\infty$ to the AR-quiver of the cluster tube \mathbf{C}_n which in terms of the above coordinate system maps all vertices $(i + rn, j + rn)$, for $r \in \mathbb{Z}$, to the same vertex. To distinguish it notationally we denote this vertex in the AR-quiver by $[(i, j)]$. In other words, every vertex $[(i, j)]$ in the AR-quiver of the cluster tube has infinitely many lifts $(i + rn, j + rn)$, where $r \in \mathbb{Z}$, to $\mathbb{Z}A_\infty$. In terms of arcs of the ∞ -gon this means that a vertex $[(i, j)]$ in the AR-quiver of the cluster tube has to be identified with the collection of all arcs of the ∞ -gon of the form $(i + rn, j + rn)$, where $r \in \mathbb{Z}$.

Noting that full subcategories closed under direct sums and direct summands are uniquely determined by the set of indecomposable objects they contain, the following observation follows immediately. It will be used throughout the paper without further mentioning.

Proposition 3.2. *Let \mathbf{C}_n be the cluster category associated to a tube of rank n . Then there are bijections between the following sets:*

- (i) *Subcategories \mathbf{X} of \mathbf{C}_n .*
- (ii) *Collections of arcs \mathcal{X} of the ∞ -gon which are n -periodic, i.e. for each arc $(i, j) \in \mathcal{X}$ also all arcs $(i + rn, j + rn)$ for $r \in \mathbb{Z}$ are in \mathcal{X} .*

One of the main results in the paper by K. Baur and R. Marsh [3] is a characterisation of the dimension of extension groups in the categories \mathbf{T}_n and \mathbf{C}_n in terms of the arc model on the ∞ -gon. For our purposes only the cluster tube situation matters and we will now restate the relevant results from [3].

For this, let us denote by σ the map on the arcs of the ∞ -gon mapping any arc to its shift by n vertices, i.e. $\sigma((i, j)) = (i+n, j+n)$, and hence $\sigma^r((i, j)) = (i+rn, j+rn)$ for all $r \in \mathbb{Z}$.

Proposition 3.3. ([3, thm. 3.8(c)]) *Let $\alpha = [(i, j)]$ and $\beta = [(k, \ell)]$ be indecomposable objects in the cluster tube \mathbf{C}_n . Supposing w.l.o.g. that for the lengths of the arcs we have $\ell - k \geq j - i$ we consider the following cardinalities,*

$$I^+ := |\{m \in \mathbb{Z} \mid i < k + mn < j\}| \text{ and } I^- := |\{m \in \mathbb{Z} \mid i < \ell + mn < j\}|.$$

Then we have

$$\dim \operatorname{Ext}_{\mathbf{C}_n}^1(\alpha, \beta) = I^+ + I^- = \dim \operatorname{Ext}_{\mathbf{C}_n}^1(\beta, \alpha).$$

Remark 3.4. (1) The cluster tube \mathbf{C}_n is a 2-Calabi-Yau category and hence the functor $\operatorname{Ext}_{\mathbf{C}_n}^1$ is symmetric in the two arguments, up to vector space duality. In particular, the dimensions occurring in the above proposition agree and the assumption $\ell - k \geq j - i$ there (and in [3, thm. 3.8(c)]) is not a restriction.

(2) Note that $I^+ + I^- > 0$ if and only if some shifted arc $\sigma^m((k, \ell))$ of the ∞ -gon crosses the arc (i, j) . (This uses the assumption that $\ell - k \geq j - i$.)

(3) An indecomposable object $\alpha = [(i, j)]$ in \mathbf{C}_n is rigid (i.e. $\operatorname{Ext}_{\mathbf{C}_n}^1(\alpha, \alpha) = 0$) if and only if $j - i \leq n$. In fact, this is exactly the condition for which no shift of α by a multiple of n crosses α .

We now characterise Ext^1 -vanishing in the cluster tube \mathbf{C}_n combinatorially in terms of the arc model.

For any collection \mathcal{X} of arcs of the ∞ -gon we denote by $\operatorname{nc} \mathcal{X}$ the collection of all arcs of the ∞ -gon which do not cross any arc from \mathcal{X} .

Proposition 3.5. *Let \mathbf{X} be a subcategory of \mathbf{C}_n and let \mathcal{X} be the corresponding n -periodic collection of arcs of the ∞ -gon. For any indecomposable object $y \in \mathbf{C}_n$ and corresponding set of arcs \bar{y} of the ∞ -gon the following are equivalent:*

- (i) $\operatorname{Ext}_{\mathbf{C}_n}^1(x, y) = 0$ for all $x \in \mathbf{X}$.
- (ii) $\bar{y} \subseteq \operatorname{nc} \mathcal{X}$.

In other words, the collection of arcs $\operatorname{nc} \mathcal{X}$ of the ∞ -gon corresponds to the subcategory $\{y \in \mathbf{C}_n \mid \operatorname{Ext}_{\mathbf{C}_n}^1(x, y) = 0 \text{ for all } x \in \mathbf{X}\}$ of \mathbf{C}_n .

Proof. First let $y \in \mathbf{C}_n$ be indecomposable such that $\operatorname{Ext}_{\mathbf{C}_n}^1(x, y) = 0$ for all $x \in \mathbf{X}$, and let \bar{y} be the corresponding set of arcs (all shifts of one of the arcs by multiples of n). Then by Proposition 3.3 and the remark following it we have that no arc from \bar{y} crosses any shift of any arc from \mathcal{X} . In particular, it does not cross any arc from \mathcal{X} itself, i.e. $\bar{y} \subseteq \operatorname{nc} \mathcal{X}$.

Conversely, let $\bar{y} \subseteq \operatorname{nc} \mathcal{X}$ be the set of arcs corresponding to an indecomposable object y in \mathbf{C}_n . Take any indecomposable $x \in \mathbf{X}$ with corresponding set of arcs $\bar{x} \subseteq \mathcal{X}$. Since $\bar{y} \subseteq \operatorname{nc} \mathcal{X}$ by assumption, no arc in \bar{y} crosses any arc in \bar{x} . Note that both \bar{x} and \bar{y} are n -periodic by definition, so Proposition 3.3 and Remark 3.4 imply that $\operatorname{Ext}_{\mathbf{C}_n}^1(x, y) = 0$, as claimed. \square

We recall the following definition from [17]; see also [12] for a finite version, which will be recalled in Section 5 below.

Definition 3.6. A collection \mathcal{X} of arcs of the ∞ -gon is called a *Ptolemy diagram* if the following condition is satisfied: if (i, j) and (r, s) are arcs in \mathcal{X} which cross, w.l.o.g. $i < r$, then all arcs among the pairs (i, r) , (i, s) , (r, j) , (j, s) must also be in \mathcal{X} .

Remark 3.7. For any collection \mathcal{X} of arcs of the ∞ -gon, $\text{nc } \mathcal{X}$ is a Ptolemy diagram. In fact, suppose (i, j) and (r, s) are crossing arcs in $\text{nc } \mathcal{X}$ with $i < r$. We have to show that the arcs (i, r) , (i, s) , (r, j) , (j, s) are also in $\text{nc } \mathcal{X}$. This immediately follows from the observation that any arc of the ∞ -gon crossing one of (i, r) , (i, s) , (r, j) , (j, s) must cross at least one of (i, j) and (r, s) . Since (i, j) and (r, s) are in $\text{nc } \mathcal{X}$ by assumption this means that no arc from \mathcal{X} can cross either of (i, r) , (i, s) , (r, j) , (j, s) , i.e. these are in $\text{nc } \mathcal{X}$, as claimed.

Proposition 3.8. *Let \mathbf{X} be a subcategory of the cluster tube \mathbf{C}_n , and let \mathcal{X} be the corresponding n -periodic collection of arcs of the ∞ -gon. Then the following conditions are equivalent:*

- (i) $\mathbf{X} = {}^\perp(\mathbf{X}^\perp)$;
- (ii) $\mathcal{X} = \text{nc } \text{nc } \mathcal{X}$;
- (iii) \mathcal{X} is an n -periodic Ptolemy diagram.

Proof. For the equivalence of (i) and (ii) we observe the following. By Proposition 3.5 the subcategory

$$\begin{aligned} \mathbf{X}^\perp &= \{y \in \mathbf{C}_n \mid \text{Hom}_{\mathbf{C}_n}(x, y) = 0 \text{ for all } x \in \mathbf{X}\} \\ &= \{y \in \mathbf{C}_n \mid \text{Ext}_{\mathbf{C}_n}^1(x, \Sigma^{-1}y) = 0 \text{ for all } x \in \mathbf{X}\} \end{aligned}$$

corresponds to the collection of arcs $\Sigma \text{nc } \mathcal{X}$. Similarly, ${}^\perp \mathbf{X}$ corresponds to $\Sigma^{-1} \text{nc } \mathcal{X}$. Taken together we have $\mathbf{X} = {}^\perp(\mathbf{X}^\perp)$ if and only if

$$\mathcal{X} = \Sigma^{-1} \text{nc } (\Sigma \text{nc } \mathcal{X}) = \text{nc } \text{nc } \mathcal{X}$$

where the latter equation holds because Σ and nc commute (since \mathbf{C}_n is 2-Calabi-Yau, the suspension Σ equals the AR translation τ and is thus just a shift by 1 on the AR quiver).

The implication (ii) \implies (iii) immediately follows from [17, lem. 3.17]; one just has to observe that condition (i) in [17, def. 0.3] is precisely our Ptolemy condition as given in Definition 3.6.

Conversely, suppose (iii) holds, i.e. \mathcal{X} is an n -periodic Ptolemy diagram. For deducing that $\mathcal{X} = \text{nc } \text{nc } \mathcal{X}$ we can again use [17, lem. 3.17]; hence we have to show that condition (ii) of [17, def. 0.4] does not occur in our context. In fact, let a vertex i be a left fountain and a vertex j with $j - i \geq 2$ be a right fountain of \mathcal{X} , i.e. there are infinitely many arcs in \mathcal{X} of the form (k, i) and infinitely many arcs in \mathcal{X} of the form (j, ℓ) . Among the former pick an arc (k, i) with $i - k \geq n + 1$. By n -periodicity all shifts by multiples of n of the infinitely many arcs of the form (j, ℓ) are also in \mathcal{X} . Since $i - k \geq n + 1$, there are infinitely many of the arcs (j, ℓ) for which some shift by a multiple of n crosses the arc (k, i) . But since \mathcal{X} is a Ptolemy diagram by assumption, we conclude that i must also be a right fountain. This shows that

condition (ii) of [17, def. 0.4] is obsolete and hence the implication (iii) \implies (ii) in our proposition also follows from [17, lem. 3.17]. \square

Recall from Figure 1 the coordinate system on the AR quiver of \mathbb{C}_n . If a vertex of the AR quiver has coordinates $[(i, j)]$ we call the natural number $j - i - 1$ the *level* of the vertex. Note that the mouth of the cylinder (i.e. the bottom line of the AR quiver) contains the vertices of level 1.

Proposition 3.9. *Let \mathbf{X} be a subcategory of the cluster tube \mathbb{C}_n , and let \mathcal{X} be the corresponding n -periodic collection of arcs of the ∞ -gon.*

Suppose that $\mathbf{X} = {}^\perp(\mathbf{X}^\perp)$, or equivalently that \mathcal{X} is an n -periodic Ptolemy diagram. Then precisely one of the following situations occurs:

- (i) \mathbf{X} has only finitely many indecomposable objects, all of level $\leq n - 1$, and \mathbf{X}^\perp contains infinitely many indecomposable objects.
- (ii) \mathbf{X} has infinitely many indecomposable objects, and \mathbf{X}^\perp contains only finitely many indecomposable objects, all of level $\leq n - 1$.

Proof. By Proposition 3.8 the condition $\mathbf{X} = {}^\perp(\mathbf{X}^\perp)$ is indeed equivalent to \mathcal{X} being an n -periodic Ptolemy diagram. So we can prove the proposition in terms of the Ptolemy diagrams \mathcal{X} and $\Sigma \text{nc } \mathcal{X}$ associated with the subcategories \mathbf{X} and \mathbf{X}^\perp , respectively.

(1) We first show that if \mathbf{X} (or \mathbf{X}^\perp) contains an indecomposable object of level $\geq n$ then \mathbf{X} (or \mathbf{X}^\perp) contains infinitely many indecomposable objects.

In fact, by assumption there is an arc in \mathcal{X} of the form $(i, i + kn + \ell)$ with $k \in \mathbb{N}$ and $1 \leq \ell < n$. By periodicity the arc $(i + kn, i + 2kn + \ell)$ is also in \mathcal{X} and it crosses $(i, i + kn + \ell)$ because $1 \leq \ell$. By assumption, \mathcal{X} is a Ptolemy diagram which implies that the arc $(i, i + 2kn + \ell)$ must then be in \mathcal{X} as well. Now repeat the argument starting with the arc $(i, i + 2kn + \ell) \in \mathcal{X}$; inductively this gives infinitely many arcs in \mathcal{X} . Having pairwise different lengths, no two of them are shifts by multiples of n of each other, i.e. they represent infinitely many different indecomposable objects in \mathbf{X} .

The above argument only used that \mathcal{X} is a Ptolemy diagram, so it applies verbatim also to the Ptolemy diagram $\Sigma \text{nc } \mathcal{X}$ (cf. Remark 3.7) and the corresponding subcategory \mathbf{X}^\perp .

Thus we have already shown that if one \mathbf{X} and \mathbf{X}^\perp has only finitely many indecomposable objects then these objects are all located strictly below level n in the AR quiver of \mathbb{C}_n .

(2) We now show that at least one of the subcategories \mathbf{X} and \mathbf{X}^\perp has infinitely many indecomposable objects, i.e. we claim that if \mathbf{X} (or \mathbf{X}^\perp) has only finitely many indecomposable objects then \mathbf{X}^\perp (or \mathbf{X}) has infinitely many indecomposable objects.

By Proposition 3.8 this amounts to showing for the corresponding collections of arcs that if \mathcal{X} is finite then $\Sigma \text{nc } \mathcal{X}$ contains arcs of arbitrary length. Since Σ only shifts by one vertex, the latter is equivalent to $\text{nc } \mathcal{X}$ containing arcs of arbitrary length.

Pick an arc from the finite collection \mathcal{X} of maximal length. W.l.o.g. (up to shifting) suppose its coordinates are $(0, \ell)$ with $\ell \geq 2$. The vertex ℓ of the ∞ -gon can not

be 'overarched' by an arc from \mathcal{X} , i.e. there is no arc $(i, j) \in \mathcal{X}$ with $i < \ell < j$; in fact, because $(0, \ell)$ has maximal length, an arc overarching the vertex ℓ had to cross $(0, \ell)$, but then the Ptolemy condition would yield an arc in \mathcal{X} longer than $(0, \ell)$, a contradiction.

Since the collection \mathcal{X} is n -periodic this even implies that no vertex of the ∞ -gon of the form $\ell + rn$ with $r \in \mathbb{Z}$ can be overarched by an arc from \mathcal{X} .

But this means that all the arcs $(\ell + rn, \ell + sn)$ for integers $r < s$ can not cross any arc from \mathcal{X} , i.e. are in $\text{nc } \mathcal{X}$. In particular, $\text{nc } \mathcal{X}$ contains infinitely many arcs.

Again, the above argument only used that \mathcal{X} is a Ptolemy diagram and therefore applies verbatim also to the Ptolemy diagram $\Sigma \text{nc } \mathcal{X}$, i.e. to \mathbf{X}^\perp .

(3) For completing the proof of Proposition 3.9 it remains to show that not both subcategories \mathbf{X} and \mathbf{X}^\perp can have infinitely many indecomposable objects.

Suppose that \mathbf{X} has infinitely many indecomposable objects. In particular, in the corresponding collection \mathcal{X} of arcs there is an arc of some length $\ell \geq n + 1$; w.l.o.g. (shifting) say its coordinates are $(0, \ell)$. Now take any arc of the ∞ -gon of length $\geq \ell$. This can not be in $\text{nc } \mathcal{X}$ since by n -periodicity of \mathcal{X} a shift of $(0, \ell)$ would cross the given arc since $\ell \geq n + 1$. This argument says that as soon as \mathcal{X} has an arc of length $\ell \geq n + 1$, all arcs of length $\geq \ell$ can't be in $\text{nc } \mathcal{X}$. After identifying any arc in $\text{nc } \mathcal{X}$ with all its shifts by multiples of n this means that there are only finitely many indecomposable objects in the corresponding subcategory $\Sigma^{-1} \mathbf{X}^\perp$, and hence also in \mathbf{X}^\perp , as claimed.

Again, the argument works equally well with the roles of \mathbf{X} and \mathbf{X}^\perp interchanged.

This completes the proof of Proposition 3.9. \square

4. THE CLASSIFICATION OF TORSION PAIRS

In this section we will classify the torsion pairs in the cluster tube \mathbf{C}_n , based on the arc model via n -periodic Ptolemy diagrams (cf. Propositions 3.8 and 3.9).

Recall from Proposition 2.2 (and the remarks preceding it) that every torsion pair in \mathbf{C}_n has the form $(\mathbf{X}, \mathbf{X}^\perp)$ where \mathbf{X} is a contravariantly finite subcategory.

Proposition 4.1. (a) *Let $(\mathbf{X}, \mathbf{X}^\perp)$ be a torsion pair in the cluster tube \mathbf{C}_n . Then the corresponding n -periodic collections of arcs \mathcal{X} and $\Sigma \text{nc } \mathcal{X}$ of the ∞ -gon are both Ptolemy diagrams and precisely one of the subcategories \mathbf{X} and \mathbf{X}^\perp has only finitely many indecomposable objects.*

(b) *Let \mathcal{X} be an n -periodic Ptolemy diagram of the ∞ -gon and let \mathbf{X} be the corresponding subcategory of \mathbf{C}_n . Then precisely one of the subcategories \mathbf{X} and \mathbf{X}^\perp has only finitely many indecomposable objects, and $(\mathbf{X}, \mathbf{X}^\perp)$ is a torsion pair.*

Proof. (a) Let $(\mathbf{X}, \mathbf{X}^\perp)$ be a torsion pair. By Proposition 2.2 we have $\mathbf{X} = {}^\perp(\mathbf{X}^\perp)$ and then by Proposition 3.8 the corresponding set of arcs \mathcal{X} is a Ptolemy diagram. Moreover, by Remark 3.7 we have that also $\Sigma \text{nc } \mathcal{X} = \text{nc } \Sigma \mathcal{X}$ is an n -periodic Ptolemy diagram. Finally, Proposition 3.9 says that $\mathbf{X} = {}^\perp(\mathbf{X}^\perp)$ implies that precisely one of the subcategories \mathbf{X} and \mathbf{X}^\perp has only finitely many indecomposable objects.

(b) Let \mathcal{X} be an n -periodic Ptolemy diagram of the ∞ -gon. Proposition 3.9 immediately gives that precisely one of the subcategories \mathbf{X} and \mathbf{X}^\perp has only finitely many indecomposable objects. It remains to show that $(\mathbf{X}, \mathbf{X}^\perp)$ is indeed a torsion pair. If \mathbf{X} (resp. \mathbf{X}^\perp) is the subcategory with only finitely many indecomposable objects then it is clearly functorially finite, thus in particular contravariantly finite (resp. covariantly finite). The corresponding collections of arcs \mathcal{X} (resp. $\Sigma \text{nc } \mathcal{X}$) are Ptolemy diagrams by assumption (resp. by Remark 3.7). Applying Proposition 3.8 we deduce that $\mathbf{X} = {}^\perp(\mathbf{X}^\perp)$ (resp. $\mathbf{X}^\perp = ({}^\perp(\mathbf{X}^\perp))^\perp$). Then Proposition 2.2 implies that $(\mathbf{X}, \mathbf{X}^\perp)$ (resp. $({}^\perp(\mathbf{X}^\perp), \mathbf{X}^\perp)$) is a torsion pair. Since $\mathbf{X} = {}^\perp(\mathbf{X}^\perp)$ we obtain in any case that $(\mathbf{X}, \mathbf{X}^\perp)$ is a torsion pair, as claimed. \square

As an immediate application we can give an alternative proof of the following result of A. Buan, R. Marsh and D. Vatne.

Corollary 4.2 ([8]). *The cluster tubes \mathbf{C}_n do not contain any cluster tilting objects.*

Proof. If u was a cluster tilting object in \mathbf{C}_n then $(\text{add}(u), \Sigma \text{add}(u))$ would be a cluster tilting subcategory (cf. [15, sec. 2.1]). In particular, it would be a torsion pair, but one for which both subcategories $\text{add}(u)$ and $\Sigma \text{add}(u)$ would have finitely many indecomposable objects, contradicting Proposition 4.1. \square

By Proposition 4.1 the classification of torsion pairs $(\mathbf{X}, \mathbf{X}^\perp)$ in the cluster tube \mathbf{C}_n reduces to the classification of the possible halves \mathbf{X} (or \mathbf{X}^\perp) of a torsion pair containing only finitely many indecomposable objects. We also know from Proposition 3.9 that in this case all indecomposable objects are strictly below level n in the AR-quiver of \mathbf{C}_n .

We shall need the following definition.

Definition 4.3. Let (i, j) be an arc of the ∞ -gon. The *wing* $W(i, j)$ of (i, j) consists of all arcs (r, s) of the ∞ -gon such that $i \leq r \leq s \leq j$, i.e. all arcs which are *overarched* by (i, j) .

Interpreting (i, j) as a vertex $[(i, j)]$ in the AR-quiver of the cluster tube \mathbf{C}_n , the corresponding wing is denoted $W[(i, j)]$; it consists of all vertices in the region bounded by the lines from $[(i, j)]$ down to the vertices $[(i, i+2)]$ and $[(j-2, j)]$, respectively.

Then we are in the position to prove our main classification result for torsion pairs in cluster tubes.

Theorem 4.4. *There are bijections between the following sets:*

- (i) *Torsion pairs $(\mathbf{X}, \mathbf{X}^\perp)$ in the cluster tube \mathbf{C}_n such that \mathbf{X} has only finitely many indecomposable objects.*
- (ii) *n -periodic Ptolemy diagrams \mathcal{X} of the ∞ -gon such that all arcs in \mathcal{X} have length at most n .*
- (iii) *Collections $\{[(i_1, j_1)], [W_1)], \dots, [(i_r, j_r)], [W_r)]\}$ of pairs consisting of vertices $[(i_\ell, j_\ell)]$ of level $\leq n-1$ in the AR-quiver of \mathbf{C}_n and subsets $[W_\ell] \subseteq W[(i_\ell, j_\ell)]$ of their wings such that for any different $k, \ell \in \{1, \dots, r\}$ we have*

$$\Sigma W[(i_k, j_k)] \cap W[(i_\ell, j_\ell)] = \emptyset$$

and the n -periodic collection W_ℓ of arcs corresponding to $[W_\ell]$ is a Ptolemy diagram (in which every arc is overarched by some arc from the collection corresponding to $[(i_\ell, j_\ell)]$).

Proof. By Proposition 3.8 we have to determine the possible n -periodic Ptolemy diagrams \mathcal{X} corresponding to a subcategory \mathbf{X} which appears as the first half of a torsion pair $(\mathbf{X}, \mathbf{X}^\perp)$ in \mathbf{C}_n and has only finitely many indecomposable objects.

Clearly, for such a Ptolemy diagram \mathcal{X} all arcs must have length at most n (recall that the level of a vertex $[(i, j)]$ is defined as $j - i - 1$, and the corresponding arc (i, j) has length $j - i$).

Conversely, let \mathcal{X} be an n -periodic Ptolemy diagram such that all arcs have length at most n . We have to show that the corresponding subcategory \mathbf{X} is the finite half of a torsion pair. Coming from a Ptolemy diagram it is immediate from Proposition 4.1 that \mathbf{X} is one half of a torsion pair. Moreover, the bounded arc lengths clearly force \mathbf{X} to be the subcategory having only finitely many indecomposable objects (in fact, an arc of length m corresponds to an indecomposable object of level $m - 1$, and for each m there are only finitely many such indecomposable objects).

This shows that the condition (ii) classifies finite halves of torsion pairs.

For obtaining the description given in (iii) one just has to translate the arc picture from (ii) to the AR-quiver of \mathbf{C}_n . The vertices $[(i_\ell, j_\ell)]$ occurring in (iii) are the ones corresponding to those arcs (i_ℓ, j_ℓ) in \mathcal{X} which are not overarched by any other arc from \mathcal{X} . Then, using the coordinate system on the AR-quiver one observes that the condition $\Sigma W[(i_k, j_k)] \cap W[(i_\ell, j_\ell)] = \emptyset$ for any different k, ℓ just expresses the property that the corresponding arcs (i_k, j_k) and (i_ℓ, j_ℓ) do not cross (since they are not overarched by any arc from \mathcal{X} they must not cross by the Ptolemy condition). The fact that the collections W_ℓ of arcs corresponding to $[W_\ell]$ can be chosen as arbitrary n -periodic Ptolemy diagrams also has been observed above. \square

5. ENUMERATION OF TORSION PAIRS

For the enumeration of all torsion pairs in \mathbf{C}_n it is convenient to rephrase the characterisation of the collections in Theorem 4.4 (iii) as follows: torsion pairs $(\mathbf{X}, \mathbf{X}^\perp)$ in the cluster tube \mathbf{C}_n such that \mathbf{X} has only finitely many indecomposable objects are in bijection with

- (iv) Collections $\{[(i_1, i_2)], [W_1], [(i_2, i_3)], [W_2], \dots, [(i_r, i_{r+1})], [W_r]\}$ with $1 \leq i_{k+1} - i_k \leq n$ and $i_{r+1} = i_1 + n$, where $[W_k]$ is the empty set if $i_{k+1} - i_k = 1$ and $[W_k]$ is a subset of the wing $W[(i_k, i_{k+1})]$ otherwise. In the latter case, $[W_k]$ interpreted as an n -periodic collection of arcs is required to be a Ptolemy diagram in which every arc is overarched by some arc from the collection corresponding to $[(i_k, i_{k+1})]$.

Note that we have $\Sigma W[(i_k, i_{k+1})] \cap W[(i_\ell, i_{\ell+1})] = \emptyset$ for all $k \neq \ell$ and whenever $i_{k+1} - i_k \geq 2$ and $i_{\ell+1} - i_\ell \geq 2$ since the arcs corresponding to the pairs in the collections $[(i_1, i_2)], [(i_2, i_3)], \dots, [(i_r, i_{r+1})]$ do not cross.

To proceed we need a combinatorial description of the individual pairs $([(i, j)], [W])$ appearing in the characterisation (iv) above. These pairs can be interpreted as sets

of diagonals of an $(j - i + 1)$ -gon with vertices labelled clockwise from i to j and where a diagonal connecting vertices k and ℓ is present if and only if the arc (k, ℓ) is in W . In the following, the edge between vertices i and j is called the *distinguished base edge*.

With this interpretation, the pairs appearing in the characterisation (iv) above are precisely the Ptolemy diagrams on the $(j - i + 1)$ -gon studied in [12] and [16], except that there the vertices are not labelled. As shown in [12, prop. 2.5] (see also [16, sec. 1.1]) the set \mathcal{P} of all such (unlabelled) Ptolemy diagrams with distinguished base edge is the disjoint union of

- the degenerate Ptolemy diagram, consisting of two vertices and the distinguished base edge only,
- a triangle with a distinguished base edge and two Ptolemy diagrams glued along their distinguished base edges onto the other edges,
- a clique, i.e. a diagram with at least four edges and all diagonals present, with a distinguished base edge and Ptolemy diagrams glued along their distinguished base edges onto the other edges,
- an empty cell, i.e. a polygon with at least four edges without diagonals, with a distinguished base edge and Ptolemy diagrams glued along their distinguished base edges onto the other edges.

We call the vertex coming first on the base edge when going counterclockwise the *distinguished base vertex*.

As in [12, sec. 3.b.] we now apply the theory of combinatorial species together with Lagrange inversion to find a formula for the number of torsion pairs in the cluster tube \mathbf{C}_n . The necessary background can be found there, or, in more detail, in the book by Bergeron, Labelle and Leroux [4, sec. 1.3].

We turn \mathcal{P} into a combinatorial species as follows: for any set U of cardinality n let $\mathcal{P}[U]$ be the set of all Ptolemy diagrams on the $(n + 1)$ -gon whose vertices other than the base vertex are labelled (bijectively) with the elements of U . Let $\mathcal{P}(z) = \mathcal{P}(z, x, y_1, y_2)$ be the associated weighted exponential generating function, where the exponent of x (of y_1 , of y_2) records the number of triangles (cliques, empty cells respectively).

Remark 5.1. Note that there are $n!$ ways to label the vertices other than the base vertex, thus the exponential generating function $\mathcal{P}(z)$ coincides with its so-called ‘ordinary’ generating function:

$$\begin{aligned} \mathcal{P}(z) &= \sum_{n \geq 1} \sum_{P \in \mathcal{P}[\{1, \dots, n\}]} x^{\#\text{triangles in } P} y_1^{\#\text{cliques in } P} y_2^{\#\text{empty cells in } P} \frac{z^n}{n!} \\ &= \sum_{n \geq 1} \sum_{P \text{ a Ptolemy diagram on the } (n+1)\text{-gon}} x^{\#\text{triangles in } P} y_1^{\#\text{cliques in } P} y_2^{\#\text{empty cells in } P} z^n. \end{aligned}$$

As we include the degenerate diagram consisting of two vertices we have

$$\mathcal{P}(z) = z + xz^2 + (2x^2 + y_1 + y_2)z^3 + \dots$$

Let C be the *species of cycles* which for any set $U = \{u_1, \dots, u_n\}$ produces the set $C[U]$ of all (oriented) cycles with n vertices labelled u_1, \dots, u_n . There are $(n-1)!$ such labelled cycles, thus the associated exponential generating function is

$$C(z) = \sum_{n \geq 0} \#C[\{1, 2, \dots, n\}] \frac{z^n}{n!} = \sum_{n \geq 1} \frac{z^n}{n} = \log \left(\frac{1}{1-z} \right).$$

We also need the following two operations on species: $\mathcal{F} \circ \mathcal{G}$ denotes the *composition* of two species \mathcal{F} and \mathcal{G} , with associated exponential generating function $\mathcal{F}(\mathcal{G}(z))$. Intuitively, the set $(\mathcal{F} \circ \mathcal{G})[U]$ can be visualised by taking an object produced by \mathcal{F} , and replacing all its labels by objects produced by \mathcal{G} , such that the set of labels is exactly U . A precise definition is given in [12] and [4, sec. 1].

The second operation is called *pointing*: for a set of labels U let $\mathcal{F}^\bullet[U] = \{(f, u) : f \in \mathcal{F}[U], u \in U\}$. In other words, \mathcal{F}^\bullet produces all \mathcal{F} -structures with a distinguished label. The associated exponential generating function satisfies $\mathcal{F}^\bullet(z) = z\mathcal{F}'(z)$, where $\mathcal{F}'(z)$ is the derivative of $\mathcal{F}(z)$.

Lemma 5.2. *Let U be a set of cardinality n . Then $(C \circ \mathcal{P})^\bullet[U]$ is in bijection with the set of n -periodic Ptolemy diagrams of the ∞ -gon with all arcs having length at most n , where the cosets of the vertices of the ∞ -gon modulo n are labelled (bijectively) with the elements of U .*

Proof. Let $((P_1, P_2, \dots, P_r), u)$ be an element of $(C \circ \mathcal{P})^\bullet[U]$. Let s be such that u is a label of P_s and suppose that u labels the ℓ^{th} vertex of P_s when going clockwise along the border of the diagram, starting after the base vertex. Thus, if P_s is a Ptolemy diagram on the $(n_s + 1)$ -gon, we have $1 \leq \ell \leq n_s$.

We can now construct in a bijective way a collection of pairs as in the characterisation (iv) given above. First we set $i_1 = -\ell$ and $i_{k+1} = i_k + n_k$ for $1 \leq k \leq r$.

We then obtain $[W_{k+1}]$ from P_{k+s} for $k \geq 0$ (setting $P_{k+s} = P_{k+s-r}$ for $k+s > r$) by the following procedure. Beginning with $k = 0$, going clockwise along the border of the diagram P_s , starting at the base vertex, we number the vertices from $-\ell$ to $n_s - \ell$. In particular, the vertex labelled with u is assigned the number 0.

Having numbered the vertices of P_{k+s} , the last number used is also assigned to the base vertex of P_{k+s+1} . We continue the numbering in this fashion.

The indecomposable object with coordinates $[(i, j)]$, $j - i \geq 2$ is included in $[W_{k+1}]$ if and only if there is an arc from the vertex numbered i to the vertex numbered j in P_{k+s} .

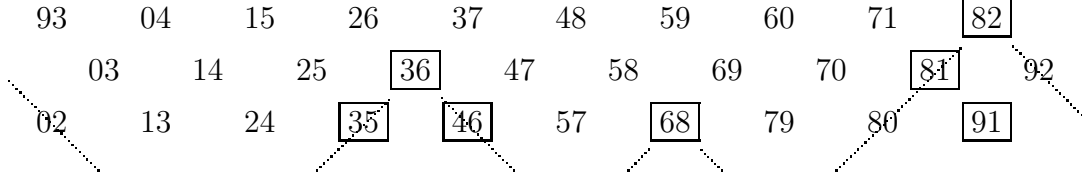
Finally we label the coset of vertices of the ∞ -gon containing i with the label of the unique non-base vertex numbered i in one of the P_k . \square

Example 5.3. Let us give an example for this bijection when $n = 10$. Consider the collection of pairs

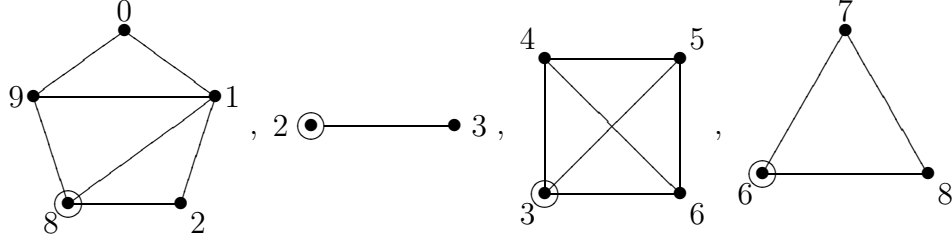
$$\begin{aligned} & \{((-2, 2), \{(8, 1), (8, 2), (9, 1)\}), \\ & \quad ((2, 3), \{\}), \\ & \quad ((3, 6), \{(3, 5), (3, 6), (4, 6)\}), \\ & \quad ((6, 8), \{(6, 8)\})\}. \end{aligned}$$

This collection satisfies the conditions in characterisation (iv).

The finite half of the corresponding torsion pair is visualised below. In this picture the vertices of the first three levels of the quiver are shown (coordinates abbreviated modulo n) and arrows are omitted. The vertices corresponding to the indecomposable objects are boxed and the wings are indicated by dotted lines (extended below level 1 to improve visibility).



Labelling vertex i of the ∞ -gon with the label $i \bmod 10$, the corresponding cycle of Ptolemy diagrams is



pointed at the vertex labelled 0.

We are now ready to prove the main theorem of this section:

Theorem 5.4. *The number of torsion pairs in the cluster tube C_n is*

$$\tau_n = \sum_{\ell \geq 0} 2^{\ell+1} \binom{n-1+\ell}{\ell} \binom{2n-1}{n-1-2\ell}.$$

More precisely, the number of torsion pairs with a total of k triangles, ℓ cliques, and m empty cells equals

$$\tau_{n,k,\ell,m} = 2 \binom{n-1+k+\ell+m}{n-1, k, \ell, m} \binom{n-1-k-\ell-m}{\ell+m}.$$

Proof. By Lemma 5.2 and Remark 5.1 we want to compute the coefficient of z^n in the exponential generating function of $(C \circ \mathcal{P})^\bullet$ which equals $z \left(\log \left(\frac{1}{1-\mathcal{P}(z)} \right) \right)'$. Since $z \left(\sum_{n \geq 0} a_n z^n \right)' = \sum_{n \geq 0} n a_n z^n$, this is the same as n times the coefficient of z^n in $\log \left(\frac{1}{1-\mathcal{P}(z)} \right)$, which we compute using Lagrange inversion.

Recall from [16, sec. 2, proof of thm. 1.3] that the generating function $\mathcal{P}(z)$ satisfies the following algebraic equation:

$$\mathcal{P}(z) = z + x\mathcal{P}(z)^2 + (y_1 + y_2) \frac{\mathcal{P}(z)^3}{1 - \mathcal{P}(z)},$$

or equivalently

$$\mathcal{P}(z) \left(1 - x\mathcal{P}(z) - (y_1 + y_2) \frac{\mathcal{P}(z)^2}{1 - \mathcal{P}(z)} \right) = z.$$

Thus $Q(z) = z(1 - xz - (y_1 + y_2) \frac{z^2}{1-z})$ is the compositional inverse of $\mathcal{P}(z)$, i.e. $Q(\mathcal{P}(z)) = z$.

Note that $(\log \frac{1}{1-z})' = \frac{1}{1-z}$. By writing $[z^n]f(z)$ for the coefficient of z^n in $f(z)$, we therefore have

$$n[z^n] \log \left(\frac{1}{1 - \mathcal{P}(z)} \right) = [z^{n-1}] \frac{1}{1-z} \frac{z^n}{Q(z)^n}.$$

Applying the multinomial theorem,

$$(1 - X - Y_1 - Y_2)^{-n} = \sum_{k, \ell, m} \binom{n-1+k+\ell+m}{n-1, k, \ell, m} X^k Y_1^\ell Y_2^m$$

we obtain by setting $X = xz$, $Y_1 = y_1 \frac{z^2}{1-z}$ and $Y_2 = y_2 \frac{z^2}{1-z}$:

$$\begin{aligned} \frac{1}{1-z} \frac{z^n}{Q(z)^n} &= \sum_{k, \ell, m} \binom{n-1+k+\ell+m}{n-1, k, \ell, m} (xz)^k y_1^\ell y_2^m \frac{z^{2(\ell+m)}}{(1-z)^{\ell+m+1}} \\ &= \sum_{k, \ell, m, i} \binom{n-1+k+\ell+m}{n-1, k, \ell, m} \binom{\ell+m+i}{\ell+m} x^k y_1^\ell y_2^m z^{i+k+2(\ell+m)}. \end{aligned}$$

To extract the coefficient of z^{n-1} we set $i = n-1-k-2(\ell+m)$. Finally, we have to multiply the result by two, since either of the two halves of a torsion pair could be the one with finitely many indecomposable objects.

To obtain the total number of torsion pairs we have to set $x = y_1 = y_2 = 1$ in $\mathcal{P}(z)$ and $Q(z)$. For the expansion of $\frac{1}{1-z} \frac{z^n}{Q(z)}$ it is then convenient to write the compositional inverse as $Q(z) = z(1-z)(1 - \frac{2z^2}{(1-z)^2})$. We leave the remaining details to the reader, which are completely analogous to the computation above. \square

Remark 5.5. Since $\mathcal{P}(z)$ satisfies an algebraic equation, so does

$$(C \circ \mathcal{P})^\bullet(z) = z \left(\log \left(\frac{1}{1 - \mathcal{P}(z)} \right) \right)' = z \frac{\mathcal{P}'(z)}{1 - \mathcal{P}(z)}.$$

Therefore, the asymptotic behaviour of its coefficients can be extracted automatically, for example using the `equivalent` function in Bruno Salvy's package `gdev` available at <http://algo.inria.fr/libraries/>. Thus, we learn that the leading term of the asymptotic expansion of $\mathcal{T}_n = [z^n]2(C \circ \mathcal{P})^\bullet(z)$ is

$$\frac{\alpha}{\sqrt{\pi n}} \rho^n,$$

where $\rho = 6.847333996370022 \dots$ is the largest positive root of $8x^3 - 48x^2 - 47x + 4$ and $\alpha = 0.2658656601482029 \dots$ is the smallest positive root of $71x^6 + 213x^4 - 72x^2 + 4$.

Because the combinatorial construction of torsion pairs is so nice, we can also compute the number of those torsion pairs that are invariant under d -fold application of Auslander-Reiten translation:

Proposition 5.6. *For any $d \in \mathbb{N}$ dividing n , the number of torsion pairs in the cluster tube \mathbf{C}_n that are invariant under d -fold application of the Auslander-Reiten translation τ equals the number of torsion pairs in the cluster tube $\mathbf{C}_{n/d}$.*

Proof. Applying τ to a torsion pair in the cluster tube \mathbf{C}_n corresponds to shifting the associated n -periodic Ptolemy diagram on the ∞ -gon by one vertex. Thus, a torsion pair invariant under τ^d corresponds to a n/d -periodic Ptolemy diagram on the ∞ -gon, which in turn corresponds to a torsion pair in the cluster tube $\mathbf{C}_{n/d}$, by Theorem 4.4. \square

Corollary 5.7. *The number of torsion pairs in the cluster tube \mathbf{C}_n up to Auslander-Reiten translation is*

$$\frac{1}{n} \sum_{d|n, d|k, d|\ell, d|m} \phi(d) \mathcal{T}_{n/d, k/d, \ell/d, m/d}.$$

Proof. For every $d|n$ there are $\phi(d)$ elements of order d in the cyclic group generated by τ , and for each group element there are $\mathcal{T}_{n/d, k/d, \ell/d, m/d}$ torsion pairs that are invariant. The corollary now follows from the Cauchy-Frobenius formula for the number of orbits,

$$\frac{1}{n} \sum_{b=0}^{n-1} \#\{\text{torsion pairs fixed by } \tau^b\},$$

and Proposition 5.6. \square

Finally we remark that the enumerative results above can be phrased as a *cyclic sieving phenomenon*, which involves a finite set \mathcal{X} , a cyclic group C of order n acting on \mathcal{X} , and a polynomial $X(q)$ with non-negative integer coefficients:

Definition 5.8. The triple $(\mathcal{X}, C, X(q))$ *exhibits the cyclic sieving phenomenon* if for every $c \in C$ we have

$$X(\omega_{o(c)}) = |\mathcal{X}^c|,$$

where $o(c)$ denotes the order of $c \in C$, ω_d is a d^{th} primitive root of unity and $\mathcal{X}^c = \{x \in \mathcal{X} : c(x) = x\}$ denotes the set of fixed points of \mathcal{X} under the action of $c \in C$.

In particular, $X(1) = |\mathcal{X}|$, i.e. $X(q)$ is a q -analogue of the generating function for \mathcal{X} . For the statement of this particular instance of the cyclic sieving phenomenon, we need to define q -binomial and q -multinomial coefficients:

Definition 5.9. For $0 \leq k \leq n$ the *q -binomial coefficient* is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!},$$

where $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$ and $[n]_q = 1 + q + \cdots + q^{n-1}$. Analogously, the *q -multinomial coefficient* is

$$\begin{bmatrix} n_1 + n_2 + \dots + n_\ell \\ n_1, n_2, \dots, n_\ell \end{bmatrix}_q = \frac{[n_1 + n_2 + \dots + n_\ell]_q!}{[n_1]_q! [n_2]_q! \cdots [n_\ell]_q!},$$

where n_1, n_2, \dots, n_ℓ are non-negative integers.

Theorem 5.10. *Let $\mathcal{T}_{n,k,\ell,m}$ be the set of torsion pairs in the cluster tube \mathcal{C}_n with k triangles, ℓ cliques and m empty cells. Let τ be the action of Auslander-Reiten translation, and let*

$$\mathcal{T}_{n,k,\ell,m}(q) = 2 \begin{bmatrix} n-1+k+\ell+m \\ n-1, k, \ell, m \end{bmatrix}_q \begin{bmatrix} n-1-k-\ell-m \\ \ell+m \end{bmatrix}_q.$$

Then $(\mathcal{T}_{n,k,\ell,m}, \langle \tau \rangle, \mathcal{T}_{n,k,\ell,m}(q))$ exhibits the cyclic sieving phenomenon.

The proof, to be given below, is an application of the q -Lucas theorem, see, for example, [20, thm. 2.2]:

Lemma 5.11 (q -Lucas theorem). *Let ω be a primitive d^{th} root of unity and a and b non-negative integers. Then*

$$\begin{bmatrix} a \\ b \end{bmatrix}_\omega = \begin{pmatrix} \lfloor \frac{a}{d} \rfloor \\ \lfloor \frac{b}{d} \rfloor \end{pmatrix} \begin{bmatrix} a - d\lfloor \frac{a}{d} \rfloor \\ b - d\lfloor \frac{b}{d} \rfloor \end{bmatrix}_\omega.$$

In particular, if $b \equiv 0 \pmod{d}$ then

$$\begin{bmatrix} a \\ b \end{bmatrix}_\omega = \begin{pmatrix} \lfloor \frac{a}{d} \rfloor \\ \lfloor \frac{b}{d} \rfloor \end{pmatrix}.$$

Proof of Theorem 5.10. Suppose that $d|n$. We have to evaluate $\mathcal{T}_{n,k,\ell,m}(q)$ at $q = \exp(2\pi i/d)$. Suppose first that $d|k$, $d|\ell$ and $d|m$. Then, expressing the multinomial coefficient as a product of binomial coefficients and applying the q -Lucas theorem, we have

$$\begin{aligned} \mathcal{T}_{n,k,\ell,m}(q) &= 2 \begin{bmatrix} n-1+k+\ell+m \\ k \end{bmatrix}_q \begin{bmatrix} n-1+\ell+m \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1+m \\ m \end{bmatrix}_q \\ &\quad \begin{bmatrix} n-1-k-\ell-m \\ \ell+m \end{bmatrix}_q \\ &= 2 \binom{\bar{n} + \bar{k} + \bar{\ell} + \bar{m} - 1}{\bar{k}} \binom{\bar{n} + \bar{\ell} + \bar{m} - 1}{\bar{\ell}} \binom{\bar{n} + \bar{m} - 1}{\bar{m}} \\ &\quad \binom{\bar{n} - \bar{k} - \bar{\ell} - \bar{m} - 1}{\bar{\ell} + \bar{m}} \\ &= \binom{\bar{n} - 1 + \bar{k} + \bar{\ell} + \bar{m}}{\bar{n} - 1, \bar{k}, \bar{\ell}, \bar{m}} \binom{\bar{n} - \bar{k} - \bar{\ell} - \bar{m} - 1}{\bar{\ell} + \bar{m}}, \end{aligned}$$

where we put $\bar{n} = n/d$, $\bar{k} = k/d$, $\bar{\ell} = \ell/d$ and $\bar{m} = m/d$.

Let us now consider the case that d does not divide all of k , ℓ and m . We show that in this situation $\begin{bmatrix} n-1+k+\ell+m \\ n-1, k, \ell, m \end{bmatrix}_q$ vanishes. Since this expression is symmetric in k , ℓ and m it is sufficient to consider the case where d does not divide m . Furthermore, expanding the multinomial coefficient in binomial coefficients as above, it is sufficient to show that $\begin{bmatrix} n-1+m \\ m \end{bmatrix}_q$ vanishes. Suppose that $m \equiv \alpha \pmod{d}$ with $0 < \alpha < d$. Then $d\lfloor \frac{n-1+m}{d} \rfloor = n/d + \alpha$ and $d\lfloor \frac{m}{d} \rfloor = m - \alpha$. Thus, by the q -Lucas theorem we obtain

$$\begin{bmatrix} n-1+m \\ m \end{bmatrix}_q = \begin{pmatrix} \lfloor \frac{n-1+m}{d} \rfloor \\ \lfloor \frac{m}{d} \rfloor \end{pmatrix} \begin{bmatrix} \alpha - 1 \\ \alpha \end{bmatrix}_q$$

which is indeed zero. □

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